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Mathematics and Music: Relating Science to Arts?

MICHAEL BEER

1. Introduction

For many people, mathematics is an enigma. Characterised by the impression of numbers and calculations taught at school, it is often accompanied by feelings of rejection and disinterest, and it is believed to be strictly rational, abstract, cold, and soulless.

Music, on the other hand, has something to do with emotion, with feelings, and with life. It is present in all daily routines. Everyone has sung a song, pressed a key on a piano, blown into a flute, and therefore, in some sense, made music. It is something people can interact with, it is a way of expression and a part of everyone's existence.

The motivation for investigating the connections between these two apparent opposites therefore is not very obvious, and it is unclear in what aspects of both topics such a relationship could be sought. Moreover, even if some mathematical aspects in music such as rhythm and pitch are accepted, it is far more difficult to imagine any musicality in mathematics. The countability and the strong order of mathematics do not seem to coincide with an artistic pattern.

However, there are different aspects which indicate this sort of relationship. Firstly, research has proved that children playing the piano often show improved reasoning skills like those applied in solving jigsaw puzzles, playing chess, or conducting mathematical deductions (see reference 1, p. 17). Secondly, it was noticed in a particular investigation that the percentage of undergraduate students having taken a music course was about eleven percent above average amongst mathematics majors (see reference 2, p. 18). This affinity of mathematicians for music is not only a recent phenomenon, but has been mentioned previously by Bloch in 1925 (see reference 3, p. 183).

This article examines the relationship between mathematics and music from three different points of view. The first describes some ideas about harmony, tones, and tunings generated by the ancient Greeks, the second shows examples of mathematical patterns in musical compositions, and the last illuminates artistic attributes of mathematics.

It is not the intention of this article to provide a complete overview of the complex connections between these two subjects. Neither is it to give detailed explanations and reasons for the cited aspects. However, this assignment will show that mathematics and music do not form such strong opposites as they are commonly considered to do, but that there are connections and similarities between them, which may explain why some musicians like mathematics and why mathematicians frequently love music.

2. Tone and tuning: the Pythagorean perception of music

In the time of the ancient Greeks, mathematics and music were strongly connected. Music was considered as a strictly mathematical discipline, handling number relationships, ratios, and proportions. In the quadrivium (the curriculum of the Pythagorean School) music was placed

<table>
<tr><td colspan="4">Mathematics
(the study of the unchangeable)</td></tr>
<tr><td colspan="2">quantity
(the discreet)</td><td colspan="2">magnitude
(the continued)</td></tr>
<tr><td>alone
(the absolute)
Arithmetic</td><td>in relation
(the relative)
Music</td><td>at rest
(the stable)
Geometry</td><td>in motion
(the moving)
Astronomy</td></tr>
</table>

Figure 1 The quadrivium (see reference 4, p. 64).

on the same level as arithmetic, geometry, and astronomy (see figure 1). This interpretation totally neglected the creative aspects of musical performance. Music was the science of sound and harmony.

The basic notions in this context were those of consonance and dissonance. People had noticed very early on that two different notes do not always sound pleasant (consonant) when played together. Moreover, the ancient Greeks discovered that to a note with a given frequency only those other notes whose frequencies were integer multiples of the first could be properly combined. If, for example, a note of the frequency 220 Hz was played, the notes of frequencies 440 Hz, 660 Hz, 880 Hz, 1100 Hz, and so on, sounded best when played together with the first.

Furthermore, examinations of different sounds showed that these integer multiples of the base frequency always appear in a weak intensity when the basic note is played. If a string whose length defines a frequency of 220 Hz is vibrating, the generated sound also contains components of the frequencies 440 Hz, 660 Hz, 880 Hz, 1100 Hz, and so on. Whereas the listeners perceive mainly the basic note, the intensities of these so-called overtones define the character of an instrument. It is primarily due to this phenomenon that a violin and a trumpet do not sound similar even if they play the same note. (The respective intensities of the overtones are expressed by the Fourier coefficients when analysing a single note played. This concept, however, will not be explained within the scope of this article.)

The most important frequency ratio is $1:2$, which is called an octave in the Western system of music notation. Two different notes in such a relation are often considered as principally the same (and are therefore given the same name), only varying in their pitch but not in their character. The Greeks saw in the octave a ‘cyclic identity’. The following ratios build the musical fifth $(2:3)$, fourth $(3:4)$, major third $(4:5)$, and minor third $(5:6)$, which all have their importance in the creation of chords. The difference between a fifth and a fourth was defined as a ‘whole’ tone, which results in a ratio of $8:9$. These ratios correspond not only to the sounding frequencies but also to the relative string lengths, which made it easy to find consonant notes starting from a base frequency. Shortening a string to two thirds of its length creates the musical interval of a fifth for example.

All these studies of ‘harmonic’ ratios and proportions were the essence of music during Pythagorean times. This perception, however, lost its importance at the end of the Middle Ages, when more complex music was developed. Despite the ‘perfect’ ratios, there occurred new dissonances when particular chords, different keys, or a greater scale of notes were used. The explanation for this phenomenon was the incommensurability of thirds, fifths, and octaves when defined by integer ratios. By adding several intervals of these types to a base note, we never reach an octave of the base note again. In other words, an octave $(1:2)$ cannot be subdivided into a finite number of equal intervals of this Pythagorean type $(x : x+1 \mid x$ being an integer$)$. Adding whole tones defined by the ratio $9:8$ to a base note with the frequency f, for example, never creates a new note with the frequency $2f$, $3f$, $4f$, or similar. However, adding six whole tones to a note almost creates its first octave defined by the following double frequency:

$$\left(\tfrac{9}{8}\right)^6 f \approx 2.0273f > 2f.$$

The amount six whole tones overpass an octave is called the 'Pythagorean comma':

$$\frac{\left(\frac{9}{8}\right)^6}{2} = 1.013\,643\,2\ldots.$$

Considering these characteristics of the Pythagorean intervals, the need for another tuning system developed. Several attempts were made, but only one has survived until nowadays: the system of dividing an octave into twelve equal ('even-tempered') semi-tones introduced by Johann Sebastian Bach. Founding on the ratio $1:2$ for octaves, all the other Pythagorean intervals were slightly tempered (adjusted) in order to fit into this new pattern. A whole tone no longer was defined by the ratio $9/8 = 1.125$, but by two semi-tones (each expressed by $\sqrt[12]{2}$) obtaining the numerical value $\sqrt[12]{2}\,\sqrt[12]{2} = \sqrt[6]{2} \approx 1.1225$. The even-tempered fifth then was defined by seven semi-tones and therefore slightly smaller than the Pythagorean fifth, the fourth by five semi-tones and therefore slightly bigger than the Pythagorean fourth.

The controversy within this tempering process is that the human ear still prefers the 'pure' Pythagorean intervals, whereas a tempered scale is necessary for complex chordal music. Musicians nowadays have to cope with these slight dissonances in order to tune an instrument so that it fits into this even-tempered pattern.

With the evolution of this more complicated mathematical model for tuning an instrument, and with the increased importance of musicality and performance, music and mathematics in this aspect have lost the close relationship known in ancient Greek times. As an even-tempered interval could no longer be expressed as a ratio ($\sqrt[12]{2}$ is an irrational number), the musicians learnt to tune an instrument by training their ear rather than by applying mathematical principles. Music from this point of view released itself from mathematical domination; see references 4 (pp. 36–67), 5, 6 (pp. 3–5), and 7 (Chapter 1, pp. 13–27).

3. Mathematical music: Fibonacci numbers and the golden section in musical compositions

The questions of tone and tuning are one aspect in which mathematical thoughts enter the world of music. However, music – at least in a modern perception – does not only consist of notes and harmony. More important are the changes of notes in relation to time, that is the aspect of rhythm and melody. Here again mathematical concepts are omnipresent. Not only is the symbolic musical notation in all its aspects very mathematical, but also particular arithmetic and geometric reflections can be found in musical compositions, as will be seen in the following paragraphs. (This article is not going to deal with highly sophisticated mathematical music theories as established by Xenakis (see reference 8) or Mazzola (see references 9 and 10) for example, based on an algebraic composition model or on group and topos theories respectively. These two concepts would exceed this overview of the relations between mathematics and music.)

A very interesting aspect of mathematical concepts in musical compositions is the appearance of Fibonacci numbers and the theory of the golden section. The former is an infinite sequence of integers named after Leonardo de Pisa (alias Fibonacci), a medieval mathematician. Its first two members are both 1, whereas every new member of the sequence is formed by the addition of the two preceding numbers, giving $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$. However, their most important feature in this context is that the sequence of Fibonacci ratios (that is the ratio of a Fibonacci number with its larger adjacent number) converges to a constant limit, called the golden ratio, golden proportion, or golden section, i.e. $0.618\,033\,98\ldots$.

More common is the geometric interpretation of the golden section: a division of a line into two unequal parts is called 'golden' if the relation of the length of the whole line to the length of the bigger part is the same as the relation of the length of the bigger part to the length of the smaller part. This proportion cannot only be found in geometric forms (for example the length of a diagonal related to the length of an edge in a regular pentagon), but also in nature (for example the length of the trunk in relation to the diameter of the tree for some particular trees, such as the Norway spruce; see reference 4, p. 113).

Due to its consideration as well-balanced, beautiful, and dynamic, the golden section has found various applications in the arts, especially in painting and photography, where important elements often divide a picture's length or width (or both) following the golden proportion. However, such a division is not necessarily undertaken consciously, but results from an impression of beauty and harmony.

Diverse studies have discovered that this same concept is also very common in musical compositions. The golden section – expressed by Fibonacci ratios – is either used to generate rhythmic changes or to develop a melody line (see reference 4, p. 116). Examples of deliberate applications can be found in the widely used 'Schillinger System of Musical Composition' or concretely in the first movement of Béla Bartók's piece 'Music for Strings, Percussion and Celeste', where, for instance, the climax is situated at bar 55 of 89 (see reference 11).

Furthermore, Rothwell's study (see reference 12) has revealed examples of the golden proportion in various musical periods. While the characteristics of the examined compositions varied greatly, the importance of proportional organisation was mostly similar. Important structural locations, marked by melodic, rhythmic, or dynamic events, were often discovered to divide the composition in two parts, either symmetrically or in the golden proportion.

A well-known example is the 'Hallelujah' chorus in Handel's Messiah. Whereas the whole consists of 94 measures, one of the most important events (entrance of solo trumpets: 'King of Kings') happens in measures 57 to 58, after about $\frac{8}{13}$ (!) of the whole piece. In addition to that, we can find a similar structure in both of the divisions of the whole piece. After $\frac{8}{13}$ of the first 57 measures, that is in measure 34, the entrance of the theme 'The kingdom of glory...' marks another essential point; and after $\frac{8}{13}$ of the second 37 measures, in measure 79 ('And he shall reign...'), again, the importance of the location is enforced by the appearance of solo trumpets (see reference 12, p. 89). It is hard to say whether Handel chose these locations deliberately, but at least this phenomenon outlines the importance of the golden section not only in visual but also in performing arts.

Another study (see reference 13, pp. 118–119) has shown that in almost all of Mozart's piano sonatas, the relation between the exposition and the development and recapitulation conforms to the golden proportion. Here, again, we cannot ascertain whether Mozart was conscious of his application of the golden section, even though some evidence suggests his attraction to mathematics.

It is probably less important to evaluate whether people consider mathematics when they apply or perceive a golden proportion than to notice that harmony and beauty – at least in this aspect – can be expressed by mathematical means. Fibonacci ratios in relation to the division of a composition, as well as integer ratios in relation to Pythagorean intervals, are examples of the fact that harmony can sometimes be described by numbers (even by integers) and therefore has a very mathematical aspect. This could be one way to introduce an additional idea: that beauty is inherent in mathematics.

4. Musical mathematics: reflections on an artistic aspect of mathematics

All these aspects of mathematical patterns in sound, harmony, and composition do not convincingly explain the outstanding affinity of mathematicians for music. Being a mathematician does not mean discovering numbers everywhere and enjoying only issues with strong mathematical connotations. The essential relation is therefore presumed to be found on another level.

It is noticeable that the above-mentioned affinity is not reciprocated. Musicians do not usually show the same interest for mathematics as mathematicians for music. We therefore must suppose that the decisive aspect cannot lie in arithmetic, the part of mathematics people sometimes consider to be in fact the whole subject. It is probably more the area of mathematical thinking, mind-setting, and problem-solving which creates these connections.

An example given by both Henle (see reference 2, p. 19) and Reid (see reference 5) is the omnipresence of words such as beauty, harmony, and elegance in mathematical research. Whereas musicians sometimes develop a particularly well-formed melody or apply an outstanding harmony, mathematicians often seek 'simple' and elegant proofs. Moreover, the sensations in solving a mathematical problem seem to be similar to those appearing when performing a musical work. Most important is the creative aspect, which lies within both of these disciplines.

Interesting evidence for this idea has been presented by Henle (see reference 2, p. 19), who compared the history of music with the history of mathematics based on the following three main arguments.

1. Mathematics has many of the characteristics of an art.
2. Viewed as an art, it is possible to identify artistic periods in mathematics: Renaissance, Baroque, Classical, and Romantic.
3. These periods coincide nicely and share many characteristics with the corresponding musical epochs, but *are significantly different* from those of painting and literature.

Relating to concepts such as dualism (Baroque), universality (Classical), and eternity (Romantic), Henle drew out surprising similarities between the evolution of mathematics and music.

Moreover, Henle outlined the necessity of a change in mathematical education towards a more musical style (see reference 2, p. 28).

> Students should make mathematics *together* (as in fact professional mathematicians do), not alone. [. . .] And finally, students should perform mathematics; they should *sing* mathematics and *dance* mathematics.

This would probably help people understand what mathematics really is, namely not divine, but mortal, and not law, but taste.

In spite of the highly speculative aspect within such ideas, this is probably the fundamental point of view when seeking connections between mathematics and music. It is the musicality in the mathematical way of thinking that attracts mathematicians to music. This, however, is difficult for people who are not familiar with this particular pattern of mind to comprehend. It is therefore probable – as has been stated by Reid (see reference 5) – that the degree of understanding such relationships is proportional to the observer's understanding of both mathematics and music. (In this context, we should mention some ideas of Hofstadter (see reference 14), who linked the music of J. S. Bach, the graphic art of Escher, and the mathematical theorems of Gödel in order to illuminate the nature of human thought processes. Once more, however, this would go beyond the framework of this article.)

5. Conclusion

This article has outlined three different approaches to the question of how mathematics and music relate to each other. The first showed the particular perception of music by the ancient Greeks, putting less importance on melody and movement than on tone, tuning, and static harmony. In the second, the concept of the golden section was brought into relation with number ratios and their occurrence in diverse compositions. The most fundamental approach, however, was the third, in which connections were revealed concerning the artistic aspect of the mathematical way of thinking.

It is obvious that these are only examples for investigating such a relationship and that other comparisons could be attempted (apart from those already mentioned). However, these three approaches represent probably the most often discussed concepts and ideas and are particularly suitable for providing a first impression of this topic.

Whatever links between music and mathematics exist, both of them are obviously still very different disciplines, and we should not try to impose one on the other. It would be wrong to attempt to explain all the shapes of music by mathematical means, just as there would be no sense in studying mathematics from a musicological point of view. However, it would be enriching if these relationships were introduced into mathematical education in order to release mathematics from its often too serious connotations.

It is important to show people that mathematics, in one way, is as much an art as it is a science. This probably would alter its common perception, and people would understand better its essence and its universality.

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Michael Beer *is a PhD student and research assistant at the Department of Quantitative Economics, University of Fribourg, Switzerland. This article is a slightly amended version of his essay entitled 'How do mathematics and music relate to each other?' written in October 1998 as an assignment at the East Coast College of English, Brisbane, Australia.*

Letters to the Editor

Dear Editor,

Regular polygons

I enjoyed reading about Daniel Schultz's interesting discovery in Volume 40, Number 2, p. 84. A short, albeit rather mechanical, proof of his general result can be given using complex numbers.

Put O, the centre of the regular n-sided polygon, at the origin and take its vertices to be (P_k), represented by the complex numbers $OP_k = z_k = R\omega^k$, where $\omega = \cos(2\pi/n) + \mathrm{i}\sin(2\pi/n)$ and, as usual, $P_{n+1} = P_1$. If the fixed point in the plane of the polygon is denoted by P with $OP = z$, then A'_k, the foot of the perpendicular from P to side P_kP_{k+1}, is of the form $OA'_k = z_k + \lambda(z_{k+1} - z_k)$ with the real number λ characterised by the fact that

$$\frac{z_k + \lambda(z_{k+1} - z_k) - z}{z_k - z_{k+1}}$$

is purely imaginary. Thus,

$$\frac{z_k + \lambda(z_{k+1} - z_k) - z}{z_k - z_{k+1}} + \frac{\bar{z}_k + \lambda(\bar{z}_{k+1} - \bar{z}_k) - \bar{z}}{\bar{z}_k - \bar{z}_{k+1}} = 0$$

which, on solving for λ and substituting $z_k = R\omega^k$, leads eventually to the expression

$$OA'_k = \tfrac{1}{2}z - \tfrac{1}{2}\bar{z}\omega^{2k+1} + \tfrac{1}{2}R\omega^k + \tfrac{1}{2}R\omega^{k+1}.$$

Since $\sum_{k=1}^{n} \omega^k = 0 = \sum_{k=1}^{n} \omega^{2k+1}$, it follows that

$$\frac{1}{n}\sum_{k=1}^{n} OA'_k = \frac{1}{2}z,$$

which proves David Wells' observation in Daniel's letter.